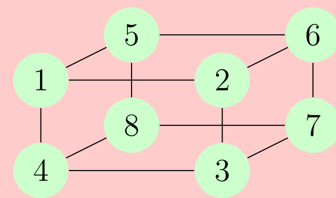
MATH 579: Combinatorics
Exam 6 Solutions

The first two questions concern the cube graph Q_3:



1. Determine, with adequate justification, whether or not Q_3 is Eulerian and/or Hamiltonian.

   For a connected graph to be Eulerian, by our theorem (proved in the HW), each vertex would have to have even degree. In Q_3, all eight vertices have odd degree. Hence it is not Eulerian. Q_3 is Hamiltonian, as proved by the following Hamiltonian cycle: $1-2-3-4-8-7-6-5-1$

2. Determine, with proof, whether or not Q_3 is bipartite.

   Q_3 is bipartite. One way to partition the vertices is: $R = \{1, 3, 6, 8\}$ and $B = \{2, 4, 5, 7\}$. We see that each face of the cube has red vertices on two opposite corners, and blue on the other two corners. Hence, each edge is between a red vertex and a blue vertex.

3. Let G be a connected (finite, simple) graph. Prove that G is a tree if and only if removing any edge of G leaves G disconnected.

   First direction: Suppose that G is a tree, and $e = \{u, v\}$ is an arbitrary edge of G. Let G' be G, with edge e removed. If u, v had some path $u - e_1 - \cdots - e_k - v$ between them in G', then $u - e_1 - \cdots - e_k - v - e - u$ would be a cycle in G. Since G is a tree, it has no cycles, so there is no such path, and hence G' is disconnected.

   Second direction: Suppose that removing any edge of G leaves it disconnected. Arguing by contradiction, suppose that G has a cycle $u - e_1 - v - \underbrace{e_2 - \cdots - e_k}_{path1} - u$. We now remove edge e_1, leaving graph G', which must be disconnected by hypothesis. Hence, there must be some vertices a, b which were connected by a path in G, but are no longer so connected in G'. Since the only difference is e_1, that edge must have been in the path connecting them in G, i.e. $\underbrace{a - \cdots - v}_{path2} - e_1 - \underbrace{u - \cdots - b}_{path3}$ (possibly with u, v reversed). But we have $\underbrace{a - \cdots - v}_{path2} - \underbrace{e_2 - \cdots - e_k}_{path1} - \underbrace{u - \cdots - b}_{path3}$, a path connecting a, b in G', a contradiction.

4. Let G be a graph. Prove that G is bipartite if and only if it contains no odd cycle.

   First direction: Suppose that G is bipartite. Any cycle in a bipartite graph must alternate vertices between the two parts, hence must have an even number of vertices, hence must be an even cycle.

   Second direction: Suppose that G contains no odd cycle; we will prove that G is bipartite. Induction on $n = |V|$. If $n = 1$, then the graph is bipartite. Suppose now that every graph of size up to n with no odd cycle must be bipartite, and we have a graph G of size $n + 1$ with no odd cycle. Choose any vertex v of G, and set G' to be the subgraph of G that removes vertex v and all its incident edges. G' must have no odd cycle (since that would be an odd cycle in G), so by the inductive hypothesis G' must be bipartite. Consider all of the vertices of G' that are adjacent to v in G; call this set N. If two of those vertices $r, s \in N$ are connected in G' but in opposite parts, then the path between them in G' must alternate parts and hence be of odd length. Hence, by adding the edges $\{r, v\}$ and $\{r, s\}$, we get an odd cycle in G, a contradiction. If instead two vertices $r, s \in N$ are of opposite parts but are not connected, then we may swap the parts of all the vertices in the connected component of s. By repeating this step as needed, we may ensure that all the vertices of N are in the same part. Now, we put v in the other part.

5. Let G be a graph. Prove that it is connected if and only if it has a spanning tree.

First direction: Suppose that G has a spanning tree T. T is connected (being a tree), and includes all the vertices of G. Hence, each pair of vertices of G is connected by a path in T, so G is connected.

Second direction: Suppose that G is connected. Proof by induction on $n = |V|$. If $n = 1$, then the sole vertex is a spanning tree. Suppose now that all connected graphs of size up to n have a spanning tree, and that G has size $n + 1$. Choose any vertex v of G, and set G' to be the subgraph of G that removes vertex v and all its incident edges. Now, G' might no longer be connected, but it does have connected components $G'_1, G'_2, \ldots, G'_k$, each of which has size no more than n. Further, each component must have at least one edge connecting to v (else removing v would not separate it). Hence, by the inductive hypothesis repeatedly, there are spanning trees T_1 for G'_1, T_2 for G'_2, $\ldots$, T_k for G'_k. Set T to be the union of $T_1, T_2, \ldots T_k$, together with v, and edges $e_1, e_2, \ldots, e_k$ (one to each G'_i component). T contains all vertices of G, so we prove T is a tree. If T has a cycle without v, that would mean that some T_i has this cycle, a contradiction. Hence, T has a cycle with v, which we may write as $v - e_i - u - \cdots - w - e_j - v$. If $i \neq j$, then the path somehow gets from component G'_i to G'_j without passing through v, which is impossible. If instead $i = j$, then $u = w$ and T_i has a cycle (from u to $w = u$), also impossible since T_i is a tree. Hence T has no cycle and is a tree.

6. Let G be a graph with n vertices and m edges. Prove that G has at least $m - n + 1$ cycles.

Let n be fixed. We proceed by induction on m. Base case: $m = n - 1$ (or less). Then, $m - n + 1 \leq 0$, so the conclusion holds trivially.

Assume now that $m \geq n$, and that every graph (with n vertices) and with at most m edges has at least $m - n + 1$ cycles. Let G be a graph (with n vertices) and $m + 1$ edges. Since $m + 1 > n$, G cannot be a tree (by our theorem that a tree on n vertices has $n - 1$ edges), and therefore G has a cycle C. Let e be any edge from that cycle C. Let G' be the subgraph of G that removes edge e. G' has m edges, hence by the inductive hypothesis has at least $m - n + 1$ cycles. However, G' does not have cycle C, since it doesn't contain edge e. Hence, G has all the $m - n + 1$ cycles that G' has, and also cycle C, for a total of at least $(m - n + 1) + 1 = (m + 1) - n + 1$ cycles.

Note that the induction stops when $m = \binom{n}{2}$, since at that point we have a complete graph and cannot add any more edges. This doesn't cause any problems for our proof.